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Extreme Points of the Set of Bistochastic Operators

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Abstract

In this article, we study the structure of the set of all quadratic bistochastic operators, as well as a complete description of all the extreme points of the set of quadratic operators. A necessary and sufficient condition for the bistochasticity of a quadratic stochastic operator is found. An analogue of Birkhoff's theorem on the extreme points of a set of bistochastic matrices is studied. A sufficient condition is found for the extremity of bistochastic quadratic operator and in a two-dimensional simplex.

Keywords: *Bistochastic quadratic operator, extreme points of the set of bistochastic matrices.*

1. Introduction

In [1], a class of quadratic stochastic operators (QSO) mapping a finite-dimensional simplex into themselves is distinguished, which are called bistochastic quadratic operators (b.q.o.). b.q.o. are closely related to the concept of majorization and. are used not only in problems of population genetics [1], [10], but also in the tasks of the economy [11]. In mathematical economics, b.q.o. is called the welfare operator. The study of b.q.o. was started in [1] and a necessary and sufficient condition for operator bistochasticity was obtained. We will give this theorem below and will use it repeatedly. In this paper, it is shown that the set b.q.o. is a convex polyhedron. Therefore, an analogue of Birkhoff's theorem on the extreme points of a set of bistochastic matrices is of interest [9]. Necessary and sufficient conditions for bistochasticity are obtained. It is observed that the set of all bistochastic quadratic operators acting in forms a polyhedron. Moreover, the number of extreme points of the set b.q.o. in a two-dimensional simplex is determined.

1. Majorization in a finite-dimensional simplex and its properties

Let $S^{n-1}=\left\{x=(x_1,...,x_n):\ \sum_{i=1}^{n}x_i=1,\ \ x_i\geq 0\right\}$ be a standard simplex in R^n. Following [11], we denote by $x_{\downarrow}=(x_{[1]},\ldots,x_{[n]})$, where $x_{[1]}\geq\ldots\geq x_{[n]}$ are the coordinates of the point ordered by non–growth.

Definition 1. [11]. If $x, y \in S^{n-1}$ and the inequalities are fulfilled $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$, $k = \overline{1,n}$, then they say that y majorizes x and write $x \prec y$.

These terms and designations were introduced by Hardy, Littlewood and Poya [12]. Obviously, for any $x \in S^{n-1}$ we have

$\left(\frac{1}{n},....,\frac{1}{n}\right) \prec x \prec (1,0,...,0)$.

As is known [11], a necessary and sufficient condition for the fulfillment of the relation $x \prec y$ is the existence of a bistochastic matrix P such that $x = Py$. Recall that a matrix $P = (P_{ij})_{i,j=\overline{1,n}}$ is called stochastic (doubly stochastic, doubly stochastic) if $P_{ij} \ge 0$, $\sum_{i=1}^{n} P_{ij} = \sum_{j=1}^{n} P_{ij} = 1$.

Therefore, for a stochastic matrix P the relation $Px \prec x$ is valid for any point $x \in S^{n-1}$. The latter property can be taken as a definition of the bistochasticity of P [11]. Following this, we will retain the term bistochasticity for an arbitrary continuous (generally speaking, nonlinear) operator $V: S^{n-1} \to S^{n-1}$, satisfying the condition

$$Vx \prec x. \qquad (1)$$

for all $x \in S^{n-1}$.

In particular, the quadratic stochastic operator $V: S^{n-1} \to S^{n-1}$ is defined by the equality

$$(Vx)_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j \ , \ k = \overline{1,n}, \qquad (2)$$

where $P_{ij,k} = P_{ji,k} \ge 0$ и $\sum_{k=1}^{n} P_{ij,k} = 1$, and when executing (1) is called bistochastic (hereinafter b.q.o.).

2. Bistochastic quadratic operators

The purpose of this paragraph is to describe the class b.q.o., which we denote by $\mathbf{B}$.

Theorem 1. [3] If $V: S^{n-1} \to S^{n-1}$ – b.k.o., then the coefficients $P_{ij,k}$ satisfy the following conditions:

$$\text{а)} \ \sum_{i,j=1}^{n} P_{ij,k} = n, \qquad \forall \ k = \overline{1,n} \qquad (3)$$

$$\text{б)} \ \sum_{j=1}^{n} P_{ij,k} \ge \frac{1}{2}, \qquad \forall \ i,k = \overline{1,n} \qquad (4)$$

$$\text{в)} \ \sum_{i,j \in I_t} P_{ij,k} \le t, \qquad \forall \ t,k = \overline{1,n}, \qquad (5)$$

where $I_t = \{i_1, i_2, ..., i_t\}$ is an arbitrary subset of the set $I = \{1, 2, ..., n\}$ containing t elements.

A matrix $T = (t_{ij})$, $i, j = \overline{1,n}$ is called *substochastic* if $t_{ij} \ge 0$ and $\sum_{j=1}^{n} t_{ij} \le 1$. If in the last inequality it has an equal sign for all $i = \overline{1,n}$ then T it is called stochastic.

Consider the following equation with respect to T.

$$A = \frac{1}{2}(T + T'), \qquad (6)$$

where T' is the transposed matrix.

Below we will deal with the solvability condition of equation (6) in the class of substochastic matrices.

Let G_n be a group of index permutations $\{1,...,n\}$. For $\pi \in G_n$, by A_π denote the matrix $A_\pi = (a_{\pi(i)\pi(j)})$, $i,j=\overline{1,n}$, which is called a permutation of the matrix A. The following sentences follow from the definitions:

- $(A_\pi)_{\pi^{-1}} = A$;
- if A is symmetric, then A_π is also symmetric;
- if A is substochastic, then T_π is also substochastic;
- $(A+B)_\pi = A_\pi + B_\pi$, $(\lambda A)_\pi = \lambda \cdot A_\pi$.

It follows from the listed properties that if T is a solution of equation (6), then T_π is a solution of the following equation:

$$A_\pi = \frac{1}{2}(T+T').$$

Theorem 2. [3] Let $A=(a_{ij})$ be a symmetric non–negative $(a_{ij}) \geq 0$ matrix. For the existence of a substochastic matrix $T=(t_{ij})$ satisfying equation (6), it is necessary and sufficient that

$$\sum_{i,j \in I_k} a_{ij} \leq k \qquad (7)$$

for any $k=\overline{1,n}$ and any set k of indexes $I_k = \{i_1,...,i_k\}$.

Proof. (conducted in [3])

Consequence. If $A=(a_{ij})$ is a non–negative matrix, then for the existence of a stochastic matrix $T=(t_{ij})$, such that $A=\frac{1}{2}(T+T')$, it is necessary and sufficient that for any set $k=\overline{1,n}$ of indices $I_k = \{i_1,...,i_k\}$ the inequality holds

$$\sum_{i,j \in I_k} a_{ij} \leq k, \qquad (8)$$

moreover, if $k=n$, then in (8) there is equality.

Remark. The substochastic matrix T, whose existence is guaranteed by Theorem 2, is not, in general, unique.

E x a m p l e. For the matrix

$$A = \begin{pmatrix} 0,1 & 0,3 & 0,4 \\ 0,3 & 0,1 & 0,5 \\ 0,4 & 0,5 & 0,4 \end{pmatrix}.$$

the set of solutions of the equation $A=\frac{1}{2}(T+T')$ in the class of substochastic matrices is

$$T_\alpha = \begin{pmatrix} 0,1 & \alpha & 0,9-\alpha \\ 0,6-\alpha & 0,1 & 0,3+\alpha \\ \alpha-0,1 & 0,7-\alpha & 0,4 \end{pmatrix},$$

where $0,1 \leq \alpha \leq 0,6$.

It is easy to see that the set of solutions to the equation $A=\frac{1}{2}(T+T')$ is a convex polyhedron. Indeed, the substochastic matrices $T=(t_{ij})$, $i,j=\overline{1,n}$, whose coefficients satisfy the following system of linear inequalities:

$$\begin{cases} 0 \le t_{ij} \le 1 \\ t_{ij} + t_{ji} = 2a_{ij} \\ \sum_{j=1}^{n} t_{ij} \le 1 \end{cases},$$

form a convex compact bounded by a plane.

Obviously, from $A = \frac{1}{2}(T + T')$ follows $(Ax, x) = (Tx, x)$, where $(\cdot,\cdot)$ is the scalar product R^n.

Indeed,

$$(Ax, x) = \frac{1}{2}((T + T')x, x) = \frac{1}{2}[(Tx, x) + (T'x, x)] = (Tx, x) \qquad (9)$$

We will continue the study of bistochastic quadratic operators..

According to theorem 1, if $V: S^{n-1} \to S^{n-1}$ – b.q.o., then $\sum_{i,j \in I_t} P_{ij,k} \le t$,

where $I_l = \{ i_1, ..., i_t \}$ is an arbitrary set t of indexes, and when $t = n$ there is equality. Putting $A_k = \left(P_{ij,k}\right)_{i,j=1}^{n}$ and using the corollary of Theorem 2, we find stochastic matrices T_k such that $A = \frac{1}{2}(T + T')$, $k = \overline{1,n}$.

According to (9), we have $\sum_{i,j=1}^{n} P_{ij,k} x_i x_j = (A_k x, x) = (T_k x, x)$.

So, if V is b.q.o., then we can find stochastic matrices $T_1, ..., T_n$, such that $Vx = \left((T_1x, x), (T_2x, x), ..., (T_nx, x)\right)$.

We prove the inequality: $\min_{1\le i\le n} x_i \le (Tx, x) \le \max_{1\le i\le n} x_i$, $\forall\ x \in S^{n-1}$, or in the accepted notation:

$$x_{[n]} \le (Tx, x) \le x_{[1]}, \forall\ x \in S^{n-1}. \qquad (10)$$

Here T is any stochastic matrix.

Indeed, if $t_{ij} \ge 0$ and $\sum_{j=1}^{n} t_{ij} = 1$, then $x_{[n]} \le \sum_{j=1}^{n} t_{ij} x_j \le x_{[1]}$

for anyone $x \in R^n$. In particular, when $x \in S^{n-1}$ we have

$$(Tx, x) = \sum_{i,j=1}^{n} t_{ij} x_i x_j = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} t_{ij} x_j \right).$$

Considering $x_i \ge 0$, $\sum_{i=1}^{n} x_i = 1$ and the previous inequality, we obtain the required inequality (10).

Theorem 3.[4] Let $A = (a_{ij})$ be a non–negative symmetric matrix. Then in order to fulfill the inequality $x_{[n]} \le (Ax, x) \le x_{[1]}$, for all $x \in S^{n-1}$, it is necessary and sufficient that $\sum_{i,j \in I_t} a_{ij} \le t$, for any $I_t = \{ i_1, ..., i_t \}$, where $t = \overline{1,n}$, and at $t = n$ equality takes place.

Let's introduce the notation::

$$\mathfrak{I}_k = \left\{ T = (t_{ij}),\ i, j = \overline{1,n}:\ 0 \le t_{ij} \le 1,\ \sum_{j=1}^{n} t_{ij} = k \right\}, \quad 1 \le k \le n$$

$$\mathbf{U}_k = \left\{ A = (a_{ij}):\ a_{ij} = a_{ji},\ A = \frac{1}{2}(T + T'),\ T \in \mathfrak{I}_k \right\}, \quad 1 \le k \le n$$

Next $\mathbf{U}_k$ we will call it symmetrization $\mathfrak{I}_k$.

Theorem 4.[4] If $A \in \mathbf{U}_k$, then for any $x \in S^{n-1}$ the inequality holds:
$x_{[n]} + x_{[n-1]} + ... + x_{[n-k+1]} \le (Ax, x) \le x_{[1]} + ... + x_{[k]}$.
Let's list a few simple properties of sets $\mathbf{U}_k$, that we will need. By E denote the matrix of order $n \times n$, all elements of which are 1.

Theorem 5. [4] The following sentences are true:
i) $A \in \mathbf{U}_k \Leftrightarrow E - A \in \mathbf{U}_{n-k}$, $k = \overline{1, n-1}$;
ii) $\mathbf{U}_k \cap \mathbf{U}_l = \varnothing$, при $k \ne l$;
iii) $\mathbf{U}_k = \{E\}$;
iv) $A \in \mathbf{U}_k \Rightarrow \frac{p}{k} \cdot A \in \mathbf{U}_p$, $1 \le p \le k$;
v) $\mathbf{U}_k + \mathbf{U}_l \supset \mathbf{U}_{k+l}$, $k + l \le n$;
vi) $\mathbf{U}_k$ is a polyhedron.

Let the matrix $T = (t_{ij}) \in \mathfrak{I}_k$ be the solution of the equation $A = \frac{1}{2}(T + T')$, where $A = (a_{ij})$ – is a given symmetric matrix with non–negative elements.
Obviously, the solvability of the last equation is equivalent to the existence of solutions to a system of linear inequalities:

$$\begin{cases} 0 \le t_{ij} \le 1; \quad i, j = \overline{1, n} \\ t_{ij} + t_{ji} = 2a_{ij}; \quad i > j \\ \sum_{j=1}^{n} t_{ij} = k; \quad i = \overline{1, n} \end{cases} . \tag{11}$$

3. Descriptions of the extreme points of the set of bistochastic quadratic operators.
A square matrix is called a permutation matrix (permutation matrix) if there is exactly one unit in each row and in each column, and all other elements are zeros.
Recall that a point x of a set A of some vector space is called an extreme if $A \setminus \{x\}$ is a convex set. The set of all extreme points of the set A is denoted by $extrA$.
According to Birkhoff's theorem [9],
(I) the extreme points of a convex set of bistochastic matrices are permutation matrices,
(II) the set of bistochastic matrices coincides with the closure of the convex hull of the set of permutation matrices.
Of course, if (I) is true, then (II) follows from
(II') the set of bistochastic matrices coincides with the closure of the convex hull of the set of its extreme points.
From the above definition of bistochasticity, obtaining any properties of b.q.o. is difficult.
Therefore, obtaining the necessary and sufficient properties for the bistochasticity of q.s.o. is very useful.

Theorem 6. [7] Let $V = (A_1 | A_2 | ... | A_m) \in \mathrm{B}$. If for some permutation π of any $m-1$ elements of $\{1, 2, ..., m\}$, $A_{\pi(k)} \in extr\mathbf{U}_1$, then $V \in extr\mathrm{B}$.

Theorem 7. [7] Let $V=(A_1|A_2|A_3|)\in \mathrm{B}$. $V\in extr\mathrm{B}$ if and only if at least two of the three matrices A_1,A_2,A_3 are extreme in $\mathbf{U}_1$.

Theorem 8. [7] If $A=\left(a_{ij}\right)\in extr\mathbf{U}_1$, then $a_{ii}=0\vee 1$, when $i\neq j$, $a_{ij}=0\vee 0{,}5\vee 1$.

If $m=2$ the conditions of Theorem 2 are necessary and sufficient. Also, the number of extreme points is 4 and they are:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0{,}5\\0{,}5&0\end{pmatrix}\begin{pmatrix}0&0{,}5\\0{,}5&1\end{pmatrix}$$

Denote:

$$H=\begin{pmatrix}0&0{,}5&0{,}5\\0{,}5&0&0{,}5\\0{,}5&0{,}5&0\end{pmatrix}$$

Corollary 1. If $m=3$, then $A\in extrU_1$ then and only then $a_{ii}=0\vee 1$, for $i\neq j$, $a_{ij}=0\vee 0{,}5\vee 1$ and $A\neq H$. Moreover, $|extr\mathbf{U}_1|=25$.

$$A_1=\begin{pmatrix}0&0&0{,}5\\0&0&0{,}5\\0{,}5&0{,}5&1\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0{,}5\\0&0&1\\0{,}5&1&0\end{pmatrix}\quad A_3=\begin{pmatrix}0&0&0{,}5\\0&1&0\\0{,}5&0&1\end{pmatrix}\quad A_4=\begin{pmatrix}0&0&0{,}5\\0&1&0{,}5\\0{,}5&0{,}5&0\end{pmatrix}$$

$$A_5=\begin{pmatrix}0&0&1\\0&0&0{,}5\\1&0{,}5&0\end{pmatrix}\quad A_6=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad A_7=\begin{pmatrix}0&0{,}5&0\\0{,}5&0&0{,}5\\0&0{,}5&1\end{pmatrix}\quad A_8=\begin{pmatrix}0&0{,}5&0\\0{,}5&0&1\\0&1&0\end{pmatrix}$$

$$A_9=\begin{pmatrix}0&0{,}5&0\\0{,}5&1&0\\0&0&1\end{pmatrix}\quad A_{10}=\begin{pmatrix}0&0{,}5&0\\0{,}5&1&0{,}5\\0&0{,}5&0\end{pmatrix}\quad A_{11}=\begin{pmatrix}0&0{,}5&0{,}5\\0{,}5&0&0\\0{,}5&0&1\end{pmatrix}\quad A_{12}=\begin{pmatrix}0&0{,}5&0{,}5\\0{,}5&1&0\\0{,}5&0&0\end{pmatrix}$$

$$A_{13}=\begin{pmatrix}0&0{,}5&1\\0{,}5&0&0\\1&0&0\end{pmatrix}\quad A_{14}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad A_{15}=\begin{pmatrix}0&1&0\\1&0&0{,}5\\0&0{,}5&0\end{pmatrix}\quad A_{16}=\begin{pmatrix}0&1&0{,}5\\1&0&0\\0{,}5&0&0\end{pmatrix}$$

$$A_{17}=\begin{pmatrix}1&0&0\\0&0&0{,}5\\0&0{,}5&1\end{pmatrix}\quad A_{18}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad A_{19}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad A_{20}=\begin{pmatrix}1&0&0\\0&1&0{,}5\\0&0{,}5&0\end{pmatrix}$$

$$A_{21}=\begin{pmatrix}1&0&0{,}5\\0&0&0{,}5\\0{,}5&0{,}5&0\end{pmatrix}\quad A_{22}=\begin{pmatrix}1&0&0{,}5\\0&1&0\\0{,}5&0&0\end{pmatrix}\quad A_{23}=\begin{pmatrix}1&0{,}5&0\\0{,}5&0&0\\0&0&1\end{pmatrix}\quad A_{24}=\begin{pmatrix}1&0{,}5&0\\0{,}5&0&0{,}5\\0&0{,}5&0\end{pmatrix}$$

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$$A_{25}=\begin{pmatrix}1 & 0,5 & 0,5\\ 0,5 & 0 & 0\\ 0,5 & 0 & 0\end{pmatrix}\qquad H=\begin{pmatrix}0 & 0,5 & 0,5\\ 0,5 & 0 & 0,5\\ 0,5 & 0,5 & 0\end{pmatrix}\qquad E=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

Corollary 2. By $m=3,\ |extr\mathbf{B}|=222.$

Proof. Let $V=extr\mathbf{B}$, $V=(A_1\,|\,A_2\,|\,A_3)$. It follow from Theorem 6 that $A_2,A_3\in extr\mathbf{U}_1$ up to permutation. Therefore, $A_1+A_2+A_3=E$ follows that either $A_1\in extr\mathbf{U}_1$, or $A_1=H$.

It is necessary to choose those triples of extreme points, the sum of which is equal to *E*. The number of such triples is 37.

Due to the permutation we get: $|extr\mathbf{B}|=37\times 3!=222.$

1. $A_1+A_{10}+A_{25}=\begin{cases}(Vx)_1=x_3^2+x_1x_3+x_2x_3\\ (Vx)_2=x_2^2+x_1x_2+x_2x_3\\ (Vx)_3=x_1^2+x_1x_2+x_1x_3\end{cases}$

2. $A_1+A_{12}+A_{24}=\begin{cases}(Vx)_1=x_3^2+x_1x_3+x_2x_3\\ (Vx)_2=x_2^2+x_1x_2+x_1x_3\\ (Vx)_3=x_1^2+x_1x_2+x_2x_3\end{cases}$

3. $A_1+A_{15}+A_{22}=\begin{cases}(Vx)_1=x_3^2+x_1x_3+x_2x_3\\ (Vx)_2=2x_1x_2+x_2x_3\\ (Vx)_3=x_1^2+x_2^2+x_1x_3\end{cases}$

4. $A_1+A_{16}+A_{20}=\begin{cases}(Vx)_1=x_3^2+x_1x_3+x_2x_3\\ (Vx)_2=2x_1x_2+x_1x_3\\ (Vx)_3=x_1^2+x_2^2+x_2x_3\end{cases}$

5. $A_2+A_9+A_{25}=\begin{cases}(Vx)_1=x_1x_3+2x_2x_3\\ (Vx)_2=x_2^2+x_3^2+x_1x_2\\ (Vx)_3=x_1^2+x_1x_2+x_1x_3\end{cases}$

6. $A_2+A_{12}+A_{23}=\begin{cases}(Vx)_1=x_1x_3+2x_2x_3\\ (Vx)_2=x_2^2+x_1x_2+x_1x_3\\ (Vx)_3=x_1^2+x_3^2+x_1x_2\end{cases}$

7. $A_2 + A_{14} + A_{22} = \begin{cases} (Vx)_1 = x_1x_3 + 2x_2x_3 \\ (Vx)_2 = x_3^2 + 2x_1x_2 \\ (Vx)_3 = x_1^2 + x_2^2 + x_1x_3 \end{cases}$

8. $A_2 + A_{16} + A_{19} = \begin{cases} (Vx)_1 = x_1x_3 + 2x_2x_3 \\ (Vx)_2 = 2x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_3^2 \end{cases}$

9. $A_3 + A_8 + A_{25} = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_3 \\ (Vx)_2 = x_1x_2 + 2x_2x_3 \\ (Vx)_3 = x_1^2 + x_1x_2 + x_1x_3 \end{cases}$

10. $A_3 + A_{15} + A_{21} = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_3 \\ (Vx)_2 = 2x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_1x_3 + x_2x_3 \end{cases}$

11. $A_3 + A_{16} + A_{18} = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_3 \\ (Vx)_2 = 2x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + 2x_2x_3 \end{cases}$

12. $A_3 + A_{24} + H = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_3 \\ (Vx)_2 = x_1^2 + x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$

13. $A_4 + A_7 + A_{25} = \begin{cases} (Vx)_1 = x_2^2 + x_1x_3 + x_2x_3 \\ (Vx)_2 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_1x_2 + x_1x_3 \end{cases}$

14. $A_4 + A_{11} + A_{24} = \begin{cases} (Vx)_1 = x_2^2 + x_1x_3 + x_2x_3 \\ (Vx)_2 = x_3^2 + x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_1x_2 + x_2x_3 \end{cases}$

15. $A_4 + A_{14} + A_{21} = \begin{cases} (Vx)_1 = x_2^2 + x_1x_3 + x_2x_3 \\ (Vx)_2 = x_3^2 + 2x_1x_2 \\ (Vx)_3 = x_1^2 + x_1x_3 + x_2x_3 \end{cases}$

16. $A_4 + A_{16} + A_{17} = \begin{cases} (Vx)_1 = x_2^2 + x_1x_3 + x_2x_3 \\ (Vx)_2 = 2x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_3^2 + x_2x_3 \end{cases}$

17. $$A_4 + A_{23} + H = \begin{cases} (Vx)_1 = x_2^2 + x_1x_3 + x_2x_3 \\ (Vx)_2 = x_1^2 + x_3^2 + x_1x_2 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$$

18. $$A_5 + A_9 + A_{24} = \begin{cases} (Vx)_1 = 2x_1x_3 + x_2x_3 \\ (Vx)_2 = x_2^2 + x_3^2 + x_1x_2 \\ (Vx)_3 = x_1^2 + x_1x_2 + x_2x_3 \end{cases}$$

19. $$A_5 + A_{10} + A_{23} = \begin{cases} (Vx)_1 = 2x_1x_3 + x_2x_3 \\ (Vx)_2 = x_2^2 + x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_3^2 + x_1x_2 \end{cases}$$

20. $$A_5 + A_{14} + A_{20} = \begin{cases} (Vx)_1 = 2x_1x_3 + x_2x_3 \\ (Vx)_2 = x_3^2 + 2x_1x_2 \\ (Vx)_3 = x_1^2 + x_2^2 + x_2x_3 \end{cases}$$

21. $$A_5 + A_{15} + A_{19} = \begin{cases} (Vx)_1 = 2x_1x_3 + x_2x_3 \\ (Vx)_2 = 2x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_3^2 \end{cases}$$

22. $$A_6 + A_7 + A_{24} = \begin{cases} (Vx)_1 = x_2^2 + 2x_1x_3 \\ (Vx)_2 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_1x_2 + x_2x_3 \end{cases}$$

23. $$A_6 + A_8 + A_{23} = \begin{cases} (Vx)_1 = x_2^2 + 2x_1x_3 \\ (Vx)_2 = x_1x_2 + 2x_2x_3 \\ (Vx)_3 = x_1^2 + x_3^2 + x_1x_2 \end{cases}$$

24. $$A_6 + A_{14} + A_{18} = \begin{cases} (Vx)_1 = x_2^2 + 2x_1x_3 \\ (Vx)_2 = x_3^2 + 2x_1x_2 \\ (Vx)_3 = x_1^2 + 2x_2x_3 \end{cases}$$

25. $$A_6 + A_{15} + A_{17} = \begin{cases} (Vx)_1 = x_2^2 + 2x_1x_3 \\ (Vx)_2 = 2x_1x_2 + x_2x_3 \\ (Vx)_3 = x_1^2 + x_3^2 + x_2x_3 \end{cases}$$

26. $$A_7 + A_{12} + A_{21} = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_2 = x_2^2 + x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_1x_3 + x_2x_3 \end{cases}$$

27. $$A_7 + A_{13} + A_{20} = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_2 = x_1x_2 + 2x_1x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_2x_3 \end{cases}$$

28. $$A_7 + A_{22} + H = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_2 = x_1^2 + x_2^2 + x_1x_3 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$$

29. $$A_8 + A_{11} + A_{22} = \begin{cases} (Vx)_1 = x_1x_2 + 2x_2x_3 \\ (Vx)_2 = x_3^2 + x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_1x_3 \end{cases}$$

30. $$A_8 + A_{13} + A_{19} = \begin{cases} (Vx)_1 = x_1x_2 + 2x_2x_3 \\ (Vx)_2 = x_1x_2 + 2x_1x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_3^2 \end{cases}$$

31. $$A_9 + A_{13} + A_{18} = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_2 \\ (Vx)_2 = x_1x_2 + 2x_1x_3 \\ (Vx)_3 = x_1^2 + 2x_2x_3 \end{cases}$$

32. $$A_9 + A_{21} + H = \begin{cases} (Vx)_1 = x_2^2 + x_3^2 + x_1x_2 \\ (Vx)_2 = x_1^2 + x_1x_3 + x_2x_3 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$$

33. $$A_{10} + A_{11} + A_{21} = \begin{cases} (Vx)_1 = x_2^2 + x_1x_2 + x_2x_3 \\ (Vx)_2 = x_3^2 + x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + x_1x_3 + x_2x_3 \end{cases}$$

34. $$A_{10} + A_{13} + A_{17} = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_2x_3 \\ (Vx)_2 = x_1x_2 + 2x_1x_3 \\ (Vx)_3 = x_1^2 + x_2^2 + x_2x_3 \end{cases}$$

35. $$A_{11} + A_{12} + A_{18} = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_1x_3 \\ (Vx)_2 = x_2^2 + x_1x_2 + x_1x_3 \\ (Vx)_3 = x_1^2 + 2x_2x_3 \end{cases}$$

36. $$A_{11} + A_{20} + H = \begin{cases} (Vx)_1 = x_3^2 + x_1x_2 + x_1x_3 \\ (Vx)_2 = x_1^2 + x_2^2 + x_2x_3 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$$

37. $$A_{12} + A_{17} + H = \begin{cases} (Vx)_1 = x_2^2 + x_1x_2 + x_1x_3 \\ (Vx)_2 = x_1^2 + x_3^2 + x_2x_3 \\ (Vx)_3 = x_1x_2 + x_1x_3 + x_2x_3 \end{cases}$$

Questions remain open about the number of red dots at $m=3$ and $m=4$. Apparently, the following article is devoted to the description of the number of extreme points at $m=3$ and $m=4$.

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