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On Graph Theoretical Properties of Extended Double Star Interconnection Network Topology

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Abstract

The Extended Double star (EDS) parallel interconnection network with a network controller (NC) is a two-level hybrid structure. It is a large-scale network with the Double star as its basic building block. EDS network has degree (n!+n+1) and diameter $\lfloor 3/2(n-1) \rfloor + 2k$ *where n and k are the two parameters denoting network dimension and the level of the network respectively. The extended double star network preserves all of the topological characteristics of the base or parent network, unlike the other derived networks. This large-scale network is suitable for voluminous computing and communications. The bipanconnectivity and hamiltonian properties of EDS are investigated. The Embedding of a guest graph into another host graph is a very important criteria and establishes the robustness of the later. For the current study of the topological relationship of EDS network with the two main classes of interconnections namely ring and mesh networks is attempted via embedding. Also, the EDS network satisfies the Hamiltonian properties. The upper limit of ring and mesh embedding is also estimated.*

Keywords: *Embedding, Star, Ring, Mesh*

1. Introduction

A parallel computing system consists of many processors working in tandem thanks to a shared memory space and interconnecting network, as well as the software that makes this possible. The connection network is the backbone of every parallel computing system. These days, massively parallel computing happens in numerous contexts, and "big data" is simply one of them. The big data notion places an emphasis on breaking down massive issues into manageable chunks that may then be tackled in parallel. In comparison to serial processing, it speeds up the time it takes to complete the task at hand. In this context, "multiple instructions and multiple data" (MIMD) computer architecture is the most well-known concept. Since Parallel Interconnection Networks (PIN) form the backbone of all types of parallel computing systems, many academics are experimenting with the underlying architecture. Major PINs are Hypercube [1], Crossed Cube [6], a Star graph [8], Extended Hypercube [3], Extended Star [2], and Double star [11] and Extended Double star [] to mention a few. If you need a larger number of processing nodes than a star network can provide, the EDS topology is a viable option. To improve the speed of computers, scientists have studied parallel processing methods for the previous two decades. The proliferation of visually beautiful interconnection networks is a direct result of the rising tide of parallelism and the recent wave of computing advancements [4]. The processing power of an IN grows as more processors are connected to it. This type of computer system, called parallel computing, uses many programs to simultaneously carry out computations and message transport across processors. Large-scale parallel computing is now a standard component of every modern situation, including big data. MIMD computers are built with networks of connections. A machine may transfer data from one node to a target node with a minimum amount of delay in any PIN. Distributed shared memory models can have their capabilities greatly enhanced by a more efficient message transmission or communication method. Each transmission and reception of information within a communicative entity must be treated as a separate event. Instead of being stored in the communications channel, the messages are instead kept by the senders and receivers at the end nodes. However, a memory block used as a communication device can be viewed as storage space for all information while engaging in shared memory communication. Applications that permit relatively independent functioning of processing units are often considered as candidates for message passing communication. Message passing in highly distributed computer systems has been a prospective study field for a long time. The notion of big data has also placed an emphasis on data exchange between different CPUs. A large data system's backbone is its parallel connectivity network. That's why it's so important to prioritize speed when storing and retrieving data from a database for processing. In parallel systems where routing and broadcasting are quicker and is performed at a cheap cost, then those will be a better choice for implementation purposes [17]. An essential component of any distributed computing system is the parallel connection network. The time between each step should be minimized as much as possible. It should allow for several simultaneous transfers of this type. In addition, it should be affordable in relation to the cost of the balance of the equipment in the system. Topology, routing protocol, switching protocol, and flow control mechanism are all characteristics that set a network apart. The most well-liked types of PIN are the hypercube and the star graph. The term "Big data" has become a popular catchphrase in recent years, and it has received a lot of media coverage as a result. Interconnection network continues to be mostly accepted to become the best reasonable type of parallel computing. This particular present effort is inspired form Extended Double Star (EDS) network [13]. A Extended double star system includes two star graphs together with a single Network Controller (NC), each one of $(2n!)^k + \frac{(2n!)^k - 1}{2n! - 1}$ nodes. Every node in an EDS has the same degree (n!+n+1). The embedding characteristics of interconnection networks play an important role, as they facilitate the transmission of messages in the event of faulty nodes. Acquiring a spanning broadcasting tree in the network is an integral part of the broadcasting process. The level of this particular spanning tree is proportional to the diameter of the system; therefore, it reveals the entire message transmission time from one node to the rest. Additionally, the robustness of the host system is also exposed through the embedding of many additional networks such as rings and meshes [10]. Using this embedding, our goal is to create a more robust and efficient message-passing system for processing of large amounts of data in parallel [5].

The remaining sections of the paper are organized as follows: The second section provides the definition and topological properties of EDS network. The third section describes the bipan connectivity and Hamiltonian property of the EDS network. The fourth section explains the embedding characteristics of an Extended Double star network. It explains the ring and mesh embedding of current EDS network. Finally, Section 5 brings the conclusion of the current paper.

Definition and Topological Properties of EDS

The EDS is a hierarchical topology with one NC and a fundamental module that is a DS graph, allowing the processing element to zero in on computation alone [11]. The EDS topology may be characterized by two parameters, n and k, where n is the DS dimension and k is the NC level, respectively. The movement from the outer ring to the inner ring and vice versa takes place through the leaf edge. Here, we use star notation to symbolize node addressing for the fundamental module. The star notation is the permutation of *n* bitstrings along with a binary bit for designating the ring structure.

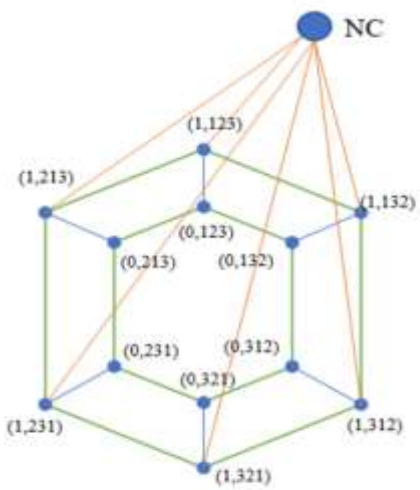

Fig. 1. Extended Double Star of Dimension Three, EDS $_{(3,1)}$

The different topological parameters of the EDS graph are discussed in detail using graph theoretical notations in [13]. The total number of computing nodes of EDS is $P = (2n!)^k + \frac{(2n!)^k - 1}{2(n!) - 1}$. The degree of

EDS network is (n!+n+1).The total number of edges in EDS network is E = $2n(n-1) + n!\,\frac{(2n!)^{k}-1}{2(n!)-1}$. The diameter of the EDS network is $\lfloor \frac{3}{2}(n-1) \rfloor + 2k$. The cost of computation in the EDS (n, k) is given by $\left[\lfloor \frac{3}{2}(n-1) \rfloor + 2k\right] \times (n! + n + 1)$.

Hamiltonian Property of EDS Network

Bipartite Property

In classical graph theory, the vertex *V* of a bipartite graph can be split into two disjoint and independent subsets $V_1$and V_2. Then there is an edge set E' subset of the edge set *E*connecting a vertex in V_1 to one in V_2. The vertex sets $V_1$and $V_2$are resulted from the vertex set *V* of the graph by removing E'. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

***Theorem1*: EDS Network is bipartite.**

Proof: In EDS apart from NC the PE_S connected to both the inner and outer rings can be split into two disjoint and independent sets of vertices V_1 andV_2 respectively. The sets are made distinct by removing the links from outer ring to inner ring. Here $V_1 \cap V_2 = Ø$. Also, in EDS network there are no odd-length cycles as each ring contains *n!*number of processing nodes. As V_1 and V_2 node sets belong to the outer and inner rings respectively, hence all removed edges are from V_1 to V_2 satisfying the bipartite condition. This implies that EDS is bipartite. (***Proved***)

Following is a simple algorithm to find out EDS graph is Bipartite

Algorithm for Bipartite EDS (n, k)

In this Bipartite EDS (n, k) algorithm there are two parameter *n* and *k*, where *n* represent the network dimension and *k* represent the level of NC. The address of the computing node is denoted as*V(x, y)*, where the*x* designates level of NC and position of NC connected to the outer ring and*y*typify the cluster address and node position inside the cluster.

Algorithm *BPEDS (V, E, n, k)*

Notations

V: Set of vertices

E: Set of edges

n: Dimension of DS

k: Level of NC

X: Denotes the level of NC and position of NC in outer ring

Y: Cluster address and node position in the cluster

E'*: Set of edges removed from original topology*

V_1*: Set of vertices in the outer ring connected to NC*

V_2*: Set of vertices in the inner ring*

1. *Start*
2. *While mod(V)>0*

{

3. *Scan v<x,y>for x address bits*
4. *(add NC to V_1)*
5. *If the node address bits is starting with 1 then Put into set V_1*
6. *Else if node address bits starting with 0 Put into set V_2.*
7. *Add the edge in E'*
8. *Scan the neighbor PE*
9. *$\|V\| = \|V\|$-1*

}

10. If $V_1 \cap V_2 = \emptyset$ and mod(V_1) =k+ mod (V_2) then Return "EDS is Bipartite"

Else Return "EDS is not Bipartite"

11. Stop

Illustration

In the Fig. 2 the removed edges are marked with red ticks depicting that EDS(3,1) is bipartite. The nodes of inner ring (node address bits starting with 0) are not connected to the NC and hence belong to node set V_2. The set of nodes with starting address bit 1 in outer ring and the NC belong to set V_1. As the base module is bipartite, hence the entire recursive structure EDS(n,k) will behave in the same manner.

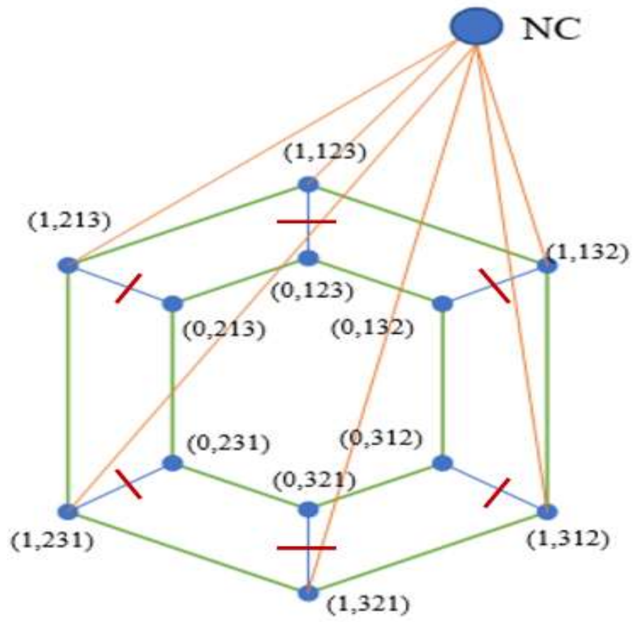

Figure 2: Bipartite Condition of Base Module EDS (3,1)

Hamiltonian Laceablility

A connected bipartite graph is called Hamilton-laceable, if it hasV_1- V_2 Hamiltonian path for all pairs of vertices $v_1$and v_2, where $v_1$belongs to one set of the bipartition, and $v_2$to the other.

Theorem 2: **EDS is Hamiltonian-laceable**

Proof: From Th.1, it's is clear that the EDS is a connected bipartite graph. It has even number of nodes. Also, EDS has V_1– V_2Hamiltonian path for all pairs of vertices $v_1$and v_2, where v_1 belongs to one set of the bipartition, and v_2 to the other. In the EDS network with the help of NC (V_1 U V_2, E') is possible where E' contains those edges that are removed to construct $V_1$and V_2. Hence EDS is Hamiltonian-lacerable. (***Proved***)

From Theorem 1 and 2 it can be very well observed that a Hamiltonian path will exist in the EDS topology. A Hamiltonian path is a path between two vertices of a graph where each vertex is visited exactly once. It is also known as a Hamilton path. In a graph, a Hamiltonian cycle can be viewed as a closed loop where the beginning and endpoints of the path are adjacent.

***Theorem3**:* **Extended Double Star $EDS_{(n,k)}$ contains a Hamiltonian cycle.**

Proof: According to the improved degree-based condition for Hamiltonicityin a graph is given by Mehedy, Kamrul and Kaykobad in 2007 as, for p number of nodes, the graph must have at least $\frac{p}{4}$ edges.

In $EDS_{(n,k)}$ topology, the count of edges (E) = $2n(n-1) + n!\left(\frac{(2n!)^k-1}{2n!-1}\right)$

In $EDS_{(n,k)}$ topology, the count of nodes (p) = $(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}$

If the total edges of $EDS_{(n,k)}$ are more than $\frac{p}{4}$ for (p > 1) then the Hamiltonian cycle must be existing.

Now, by using mathematical induction method we have to proof that:

$S_{n,k}$: $2n(n-1)+n!\left(\frac{(2n!)^k-1}{2n!-1}\right)>\frac{(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}}{4}$ is true. (for n ≥ 3, k = 1)

Base Case (show that $S_{n,k}$ is true):

For, $S_{3,1}$: $2\times 3\ (3-1)+3!\left(\frac{(2\times 3!)^1-1}{2\times 3!-1}\right)>\frac{(2\times 3!)^1+\frac{(2\times 3!)^1-1}{2(3!)-1}}{4}$

⇒$S_{3,1}$: (12 + 1) $>\frac{13}{4}$

⇒$S_{3,1}$: 13 > 4

So, the base case $S_{3,1}$ is true.

For inductive hypothesis, assume that $S_{t,r}$ is true for t ≥ 3 & r = 1

$S_{t,r}$: $2t(t-1)+t!\left(\frac{(2t!)^r-1}{2t!-1}\right)>\frac{(2t!)^r+\frac{(2t!)^r-1}{2(t!)-1}}{4}$ is true. (for t ≥ 3, r = 1)

⇒ $S_{t,r}$: $2t^2$ - $2t$ + $t!$ $>\frac{2t!+1}{4}$ is true. (for, r = 1)

We have to show that, $S_{t,r}$ is true follows that $S_{t+1,r+1}$ is true.

Consider, $S_{t+1,r+1} = 2(t+1)(t+1-1)+(t+1)!\left(\frac{\{2(t+1)!\}^{r+1}-1}{2(t+1)!-1}\right)$

$= 2t(t+1)+(t+1)!\left(\frac{\{2(t+1)!\}^2-1}{2(t+1)!-1}\right)$ (for, r = 1)

$= 2t^2+2t+(t+1)!\,\{2(\mathrm{t}+1)!\}^2+1$

$= 2t^2+2t+4\{(\mathrm{t}+1)!\}^3+1$

$= 2t^2+2t+4\{(t+1)!\}^3+1+2t - 2t+t! - t!$

$= 2t^2 - 2t+t! + 2t+4\{(t+1)!\}^3+1+2t - t!$

$>\frac{2t!+1}{4}+4t - t!+4\{(t+1)!\}^3+1$

$>\frac{2t!+1}{4}+\ 4[t+\{(t+1)!\}^3]\ -\ t!+1$

$>\frac{2(t+1)!^2+2(t+1)!+1}{4}$

$>\frac{2(t+1)!\{(t+1)!+1\}+1}{4}$ (R.H.S)

Therefore, $S_{t,r}$ is true follows that $S_{t+1,r+1}$ is true.

So, by mathematical induction method, it can be assumed that

$S_{n,k}$: $2n(n-1)+n!\left(\frac{(2n!)^k-1}{2n!-1}\right)>\frac{(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}}{4}$ is true. (for n ≥ 3, k = 1)

In this way we can prove that,

$S_{n,k}$: $2n(n-1)+n!\left(\frac{(2n!)^k-1}{2n!-1}\right)>\frac{(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}}{4}$ is true. (for n ≥ 3, k = 2)

$S_{n,k}$: $2n(n-1)+n!\left(\frac{(2n!)^k-1}{2n!-1}\right)>\frac{(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}}{4}$ is true. (for n ≥ 3, k = 3)

.

. . ….. & so, on

Therefore, $S_{n,k}$: $2n(n-1)+n!\left(\frac{(2n!)^k-1}{2n!-1}\right)>\frac{(2n!)^k+\frac{(2n!)^k-1}{2(n!)-1}}{4}$ is true. (for n ≥ 3, k ≥ 2)

This implied that, Extended Double star $EDS_{(n,k)}$ contains a Hamiltonian cycle. ***(Proved)***

Embedding

Graph embedding has been increasingly essential in a range of computer architecture and machine learning techniques in recent years. We perform multiple tasks, including clustering, principal component analysis (PCA), classification, etc., in a very robust manner using the nodes, edges and other components of the graph embedding. Embedding is a most commonly used machine learning technique that involves with the representation of a complex object such as texts, images, as well as graphs into a vector with a reduced number of features as compared to the dimension of the dataset of billions number of nodes in a graph, while still sustaining the most important information about them.

Let, G is a guest and H is a host finite graphs of n vertices, where $V(G)$and $V(H)$ denote the vertex set of $G\ andH$ and $E(G)$ and $E(H)$ denote the edge set of $G\ andH$ respectively. Then an embedding function$f$ of G into H is defined as:

1. f is a one-to-one mapping from $V(G) \to V(H)$.

2. P_f is a one-to-one mapping from $E(G)\ to\ P_f\big(f(u), f(v)\big) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)$ for $\{(u, v) \in E(G)\}$.

From the definition it is clear that the graph embedding assigns a fixed-length vector representation to each entity (typically nodes) in the graph. These embedding preserve the graph's topology, which are lower dimensional representation of the graph. Graph embedding allows for the efficient simulation of one network architecture through another. Embedding is very essential part of computer science because of we can easily modified an algorithm for graph H, which is made for graph G. Dilation, congestion, and expansion are among the parameters related to graph embedding. The dilation of an embedding is defined as the maximum length of such paths that can be taken over all source edges. Link congestion provides the most of the inter-process communication source channels that can be mapped to a single physical host link. To reduce contention on links or router buffers, link congestion should be minimized. Node congestion defines the maximum number of inter-process channels that can pass through a single host router. Expansion is the measure of processor utilization. It is defined as the ratio of the vertex size of host to guest graph.

Ring Embedding in EDS

The EDS is an interconnection network that consists oftwo-star networks linked to a single network controller (NC). Previously, studies are made how different topologies can be embedded as binary trees, meshes, rings of stars, in CQs, and SCQs. Through this method, we are able to acquire ring properties in EDS analogous to a star and double star network. A ring is usually used to describe a path where the starting and ending nodes are the same and same node cannot be travelled more than once. The Hamiltonian cycle has already been derived from an EDS network. In below, after passing through each node of the EDS network at least once, the route is finished when it reaches the node from which it initially set off.

Lemma 1: The embedding of ring is possible in EDS $_{(3,1)}$ and the upper bound on the size of the ring is 13.

Proof: By the use of gray code, we must encode numbers so that the only difference between them is a single digit. Frequently, the term Gray code refers to a "reflected" code, or more precisely, the binary reflected Gray code. Developing an n-bit Gray code in EDS:

EDS $_{(3,1)}$ network consist of two-star graphs together with single Network Controller. A ring with 12 nodes (R1 through R12) and one NC is shown in Fig.3. The addresses of the nodes in the EDS graph are given below.

R1= (1,123) R2= (1,132) R3= (1,312)

R4= (1,321) R5= (1,231) R6= (1,213)

R7= (0,213) R8= (0,231) R9= (0,321)

R10= (0,312) R11= (0,132) R12= (0,123)

R13= NC (Network Controller)

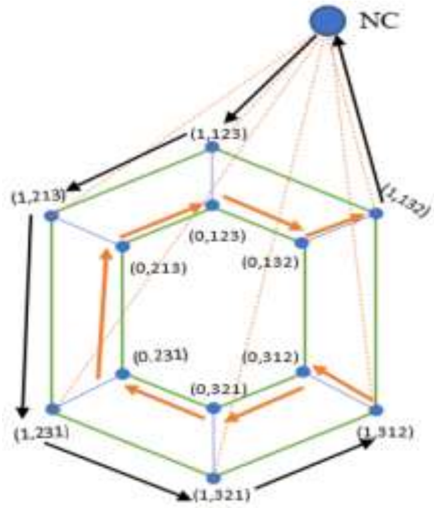

Fig. 3. Embedding of Ring Topology in EDS $_{(3,1)}$

In EDS $_{(3,1)}$ network 12 edges and 1 NC are to be considered. Therefore, 12+1= 13 number of edges needed to embed a ring in EDS $_{(3,1)}$ network shown in the Fig.3. ***(Proved)***

Lemma 2: The upper bound of the ring sizeembedded in EDS $_{(3,2)}$ network is 169.

Proof: In order to embed a ring in EDS $_{(3,2)}$, all nodes will be covered just once and the beginning and ending points being the same. Right after travelling through a cluster, depending upon the y part of the node address, next star graph neighbor will be chosen as the next subsequent related clusteras shown in Fig.4 below. The EDS $_{(3,2)}$ network consists of twelve numbers of clusters connected in star manner to the main basic block DS network in inner and outer ring as shown in Fig.4 and a single Network Controller at level 0. At level one, there are 2n! numbers of clusters and each cluster has 13 nodes (12 computing nodes and one NC). To embed a ring in each network, 13 edges are needed. Therefore, total 13 × 12 = 156 number of edges are needed to embed a ring. At level zero, the EDS $_{(3,2)}$ network consists of a DS network with a single NC, which has 13 nodes and it required total 13 edges to embed a ring. So, maximum 156 + 13 = 169 edges are needed to embed a ring in theEDS$_{(3,2)}$ network. ***(Proved)***

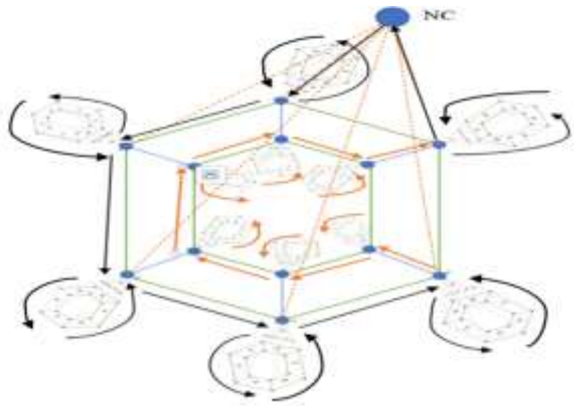

Fig. 4. Embedding of Ring topology in EDS $_{(3,2)}$

Lemma 3: The upper bound on the size of the ring possible in EDS $_{(4,1)}$ network topology is 49.

Proof: EDS $_{(4,1)}$ network consists of four number of double star networks together with single Network Controller. Each double star network needed 11 number of edges to embed a ring in it as shown in the Fig.5. So, there are total 11 × 4 = 44 number of edges are required to embed a ring in four different DS networks. Furthermore, each DS networks are connected with each other in star manner. Therefore, again five number of edges are needed to embed a complete ring shown in the Fig.5. So, maximum 44 + 5 = 49 edges are needed to embed a ring in an EDS $_{(4,1)}$ network. ***(Proved)***

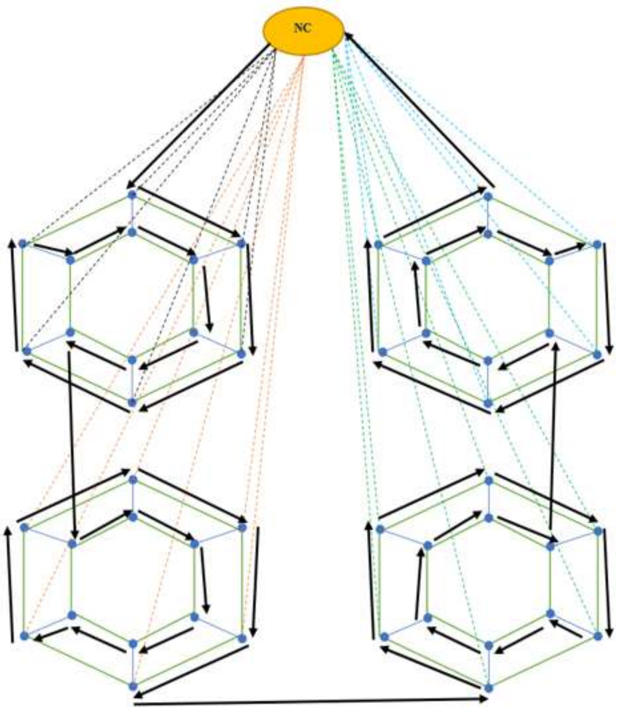

Fig. 5. Embedding of Ring topology in EDS $_{(4,1)}$

Lemma 4: The maximum size of the ring in EDS $_{(4,2)}$ network is 673.

Proof:EDS$_{(4,2)}$ network consists of four number of double star networks with a single Network Controller. Each double star networks consists of twelve numbers of clusters connecting in star manner shown in the Fig.6. At level one, each clusters have 13 nodes, which required total 13 number of edges to embed a ring and there is total four number of independent DS networks shown in Fig.6. So, there are total $(13 \times 12) \times 4 = 624$ number of edges are needed to embed ring. At level zero, each double star network required 11 number of edges to embed a ring in it as shown in the Fig.6. So, there are total $11 \times 4 = 44$ number of edges are required. Furthermore, each DS networks are connected with each other in star manner. So, again five number of edges are required to embed a complete ring shown in the Fig.6. Therefore, at level zero the network required total $44 + 5 = 49$ edges to embed a ring. So, maximum $624 + 49 = 673$ edges are needed to embed a ring in EDS $_{(4,2)}$ network. ***(Proved)***

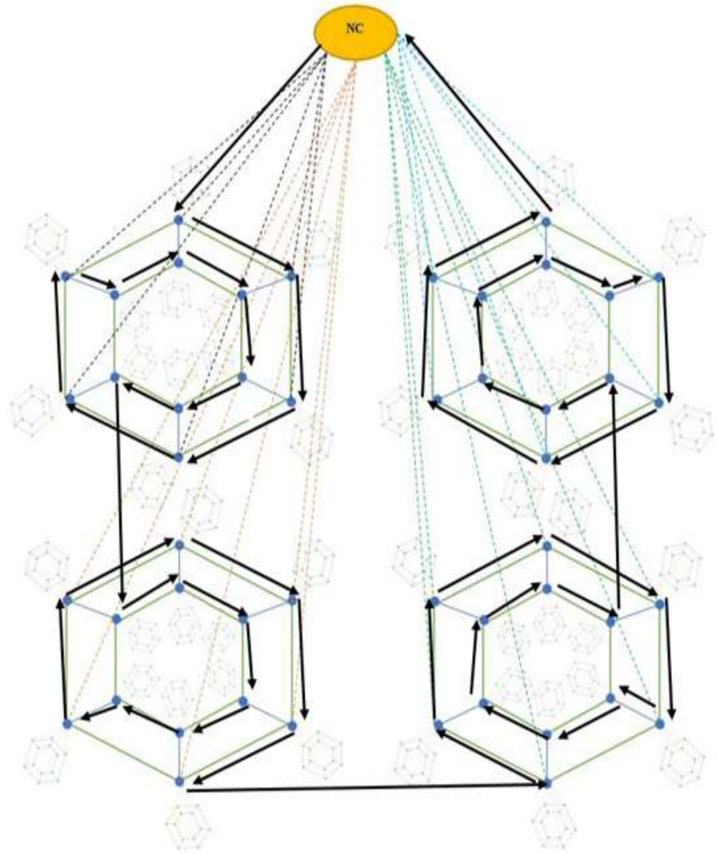

Fig. 6. Embedding of Ring topology in EDS $_{(4,2)}$

Theorem-1: **The upper bound on the ring size embedded in EDS$_{(n,k)}$ is (2n! + 1) + 52 (k-1) (n) (n-1) (n-2) …3.**

Proof: The Extended Double Star structurecontains $(2n!)^k + \frac{(2n!)^k - 1}{(2n)! - 1}$ number of computing nodes. To calculate the total sum of connecting edges in EDS$_{(n,k)}$ to embed a ring, a comparison is being made from the above stated four ring embeddings. From *Lemma-1*: In EDS $_{(3,1)}$ network, the maximum size of the ring is 13. At level zero, the EDS $_{(3,1)}$ network has 13 edges, which can be written as (2n! + 1) edges. Therefore, we can say that in general the EDS $_{(3,1)}$ network has (2n! + 1) + (k-1) edges. Where, (k-1) is the highest level of NC in EDS$_{(n,k)}$network. From *Lemma-2*: In EDS$_{(3,2)}$ network, the maximum size of the ring is 169. At level zero, the EDS $_{(3,2)}$ network has 13 edges, which can be written as (2n! + 1) edges. At level one, the EDS $_{(3,2)}$ network has (13 × 12) edges, which can be written as 52 × n edges. Therefore, we can say that in general the network has (2n! + 1) + (k-1) × 52 × n edges. Where, (k-1) is the highest level of NC in EDS$_{(n,k)}$ network. From *Lemma-3*: In EDS $_{(4,1)}$ network, the maximum size of the ring is 49. At level zero, the EDS $_{(4,1)}$ network has 49 edges, which can be written as (2n! + 1) edges. Therefore, we can say in general the network has (2n! + 1) + (k-1) edges. Where, (k-1) is the highest level of NC in EDS$_{(n,k)}$ network. From *Lemma-4*: In EDS $_{(4,2)}$ network, the maximum size of the ring is 673. At level zero, the EDS $_{(4,2)}$ network has 49 edges, which can be written as (2n! + 1) edges. At level one, the EDS $_{(4,2)}$ network has [(13 × 12) × 4] edges, which can be written as 52 × n × (n-1) edges.

Therefore, we can say in general the network has (2n! + 1) + (k-1) × 52 × n × (n-1) edges, where, (k-1) is the highest level of NC in the EDS$_{(n,k)}$ network. After considering all of the above four cases, to get a generalized equation for EDS$_{(n,k)}$, the upper bound on the size of the ring contained in EDS$_{(n,k)}$ structure will be (2n! + 1) + 52 (k-1) (n) (n-1) (n-2) ….3. ***(Proved)***

From the Theorem 1 it is clear that the dilation and expansion of ring embedding in EDS topology is one. Also, the ring embedding has unit link and node congestion.

Mesh Embedding in EDS

Mesh network is nothing but more than connect one processor to four other processors. In mesh network, there is a connection between the processors in the last column and the first processor of the next row. There should be a connection between the processors in the bottom right and top left corners.

Lemma 5: The upper bound on the number of 2D disjoint mesh embedding in EDS $_{(3,1)}$ is 6.

Proof: The EDS $_{(3,1)}$ topology consists of two sets of star graphs organized into two rings with single Network Controller connected to the outer rings. The links connecting the inner ring with the outer ring encompass six numbers of two dimensional meshes as shown in Fig.7 (a). Here the meshes shown in figure are numbered from 1 to 6 as shown in Fig. 7 (b). Hence, maximum 6 numbers of 2D meshes are contained in EDS $_{(3,1)}$. ***(Proved)***

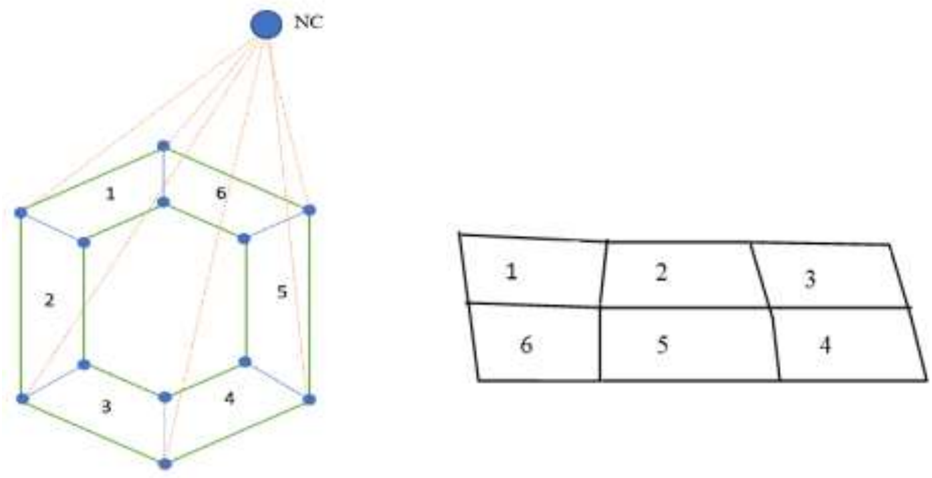

Fig. 7. (a) Mesh Embedding in EDS $_{(3,1)}$, (b) Six number of 2D Meshes in EDS $_{(3,1)}$

Lemma 6: The highest number of meshes contained in the EDS $_{(3,2)}$ topology is determined to be 78.

Proof: The EDS $_{(3,2)}$ network consists of a double star network with a single Network Controller and the double star network has twelve numbers of clusters connecting in star manner shown in Fig.8. At level one, the network has twelve number of clusters. Each clusters creates 6 number of meshes. Therefore, total (12 × 6) = 72 number of meshes are created by the network. At level zero, the EDS $_{(3,2)}$ network consists of single DS network with a NC, which embed with six number of meshes. So, maximum 72 + 6 = 78 meshes are possible in EDS $_{(3,2)}$.

(proved)

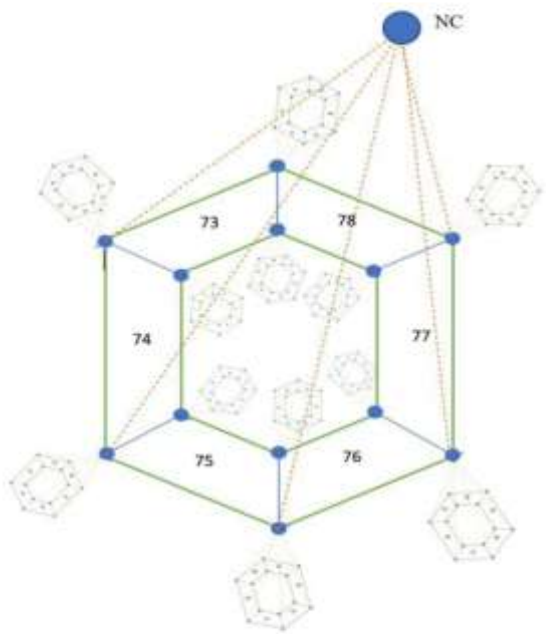

Fig. 8. Embedding of Mesh in EDS $_{(3,2)}$

Lemma 7: The highest number of meshes contained in the EDS $_{(4,1)}$ topology is determined to be 36.

Proof: EDS $_{(4,1)}$ network consists of four double star networks linked to a single Network Controller and each double star networks embedded with six number of meshes. So, total 6 × 4 = 24 number of meshes are created by the DS networks. Again, all of the four double star networks create two sets of meshes with each other shown in the Fig.9. Therefore, the EDS network creates twelve number of meshes. So, maximum 24 + 12 = 36 meshes number of meshes are possible in EDS $_{(4,1)}$. ***(Proved)***

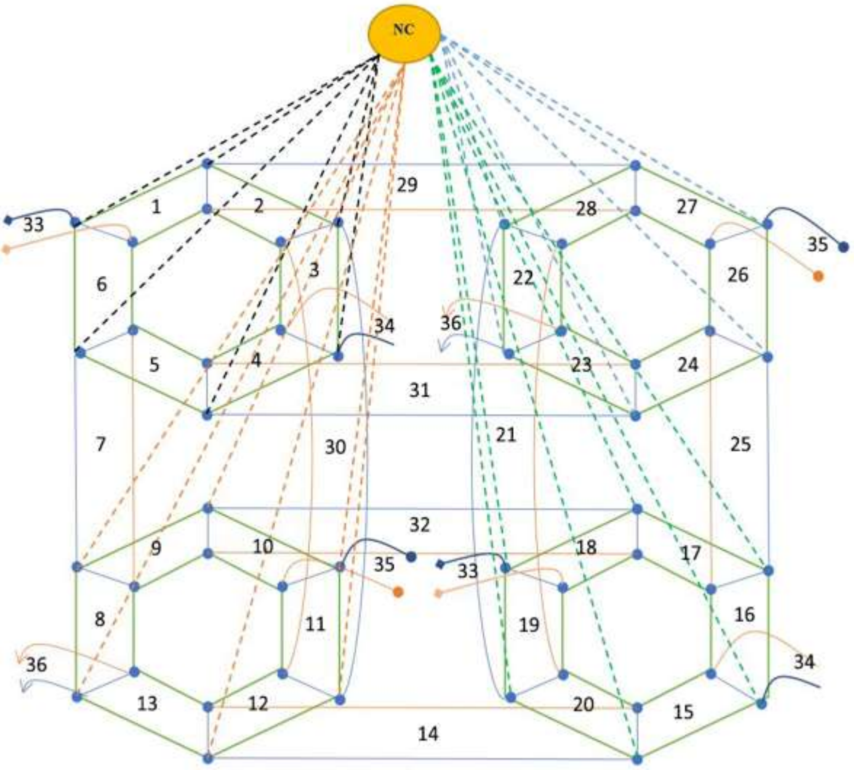

Fig. 9. Embedding of Mesh in EDS $_{(4,1)}$

Lemma 8: The upper bound on the mesh embedding into the EDS $_{(4,2)}$ topology is determined to be 324.

Proof: EDS $_{(4,2)}$ network consists of four number of double star networks with a single Network Controller and each double star networks have twelve numbers of clusters connecting in star manner shown in the Fig.10. At level one, each clusters creates six meshes and there are total (12×4) number of clusters. So, the maximum $12 \times 4 \times 6 = 288$ number of meshes possible at level one. At level zero, EDS $_{(4,2)}$ network consists of four number of double star networks which are connected to a single NC. Each double star network embedded with six number of meshes. So, total $6 \times 4 = 24$ number of meshes are created by the DS networks. Then, all of the four double star networks create two sets of meshes with each other shown in the Fig.11 and again are created 12 number of meshes. Therefore, at level zero maximum $24 + 12 = 36$ meshes number of meshes are possible. So, maximum $288 + 36 = 324$ number of meshes possible in EDS $_{(4,2)}$ network. ***(Proved)***

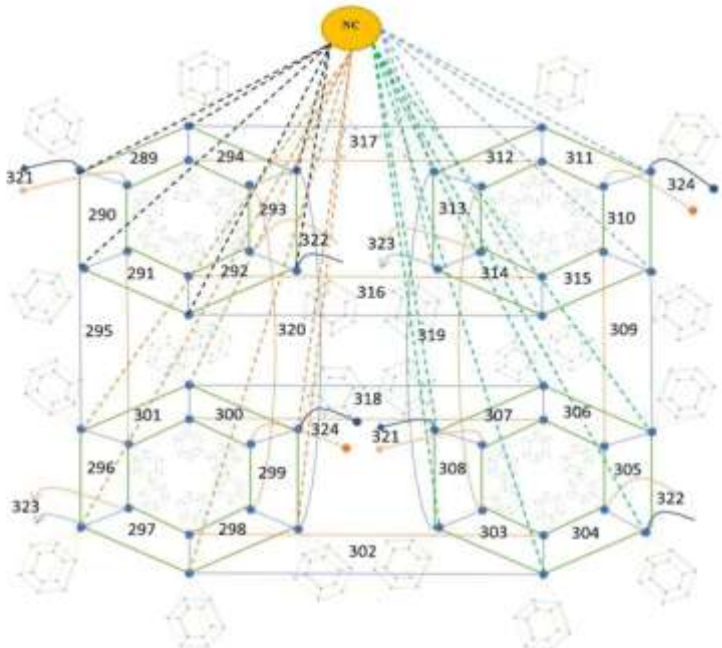

Fig. 10. Mesh Embedding in EDS $_{(4,2)}$

Theorem-2: The total number of meshes that can be embedded in $EDS_{(n,k)}$ is $\frac{n!(n-1)}{2}$ + 24 (k-1) (n) (n- 1) (n- 2)….3 .

Proof: The $EDS_{(n,k)}$ is a n dimensional network of k levels. The EDS network has $EDS_{(3,1)}$ as the basic module and the network has six number of meshes. To calculate the total number of meshes in $EDS_{(n,k)}$, we have to compare the above four mesh embedding of extended double star network. From lemma-5: In $EDS_{(3,1)}$ network, the maximum size of the meshes are six. At level zero, the $EDS_{(3,1)}$ network has six meshes, which can be written as $\frac{n!(n-1)}{2}$ meshes. Therefore, we can say that in general the network has $\frac{n!(n-1)}{2}$+ (k-1) meshes. Where, (k-1) is the highest level of NC in $EDS_{(n,k)}$ network. From lemma-6: In $EDS_{(3,2)}$ network, the maximum size of the meshes are 78. At level zero, the $EDS_{(3,2)}$ network has six meshes, which can be written as $\frac{n!(n-1)}{2}$ meshes. At level one, the $EDS_{(3,2)}$ network has (12 × 6) meshes, which can be written as 24 × n meshes. Therefore, we can say that in general the network has $\frac{n!(n-1)}{2}$ + (k-1) × 24 × n meshes. From lemma-7: In $EDS_{(4,1)}$ network, the maximum size of the meshes are 36. At level zero, the $EDS_{(4,1)}$ network has 36 meshes, which can be written as $\frac{n!(n-1)}{2}$ meshes. Therefore, we can say that in the network has $\frac{n!(n-1)}{2}$ + (k-1) meshes. From lemma-8: In $EDS_{(4,2)}$ network, the maximum size of the meshes is 324. At level zero, the $EDS_{(4,2)}$ network has 36 meshes, which can be written as $\frac{n!(n-1)}{2}$ meshes. At level one, the $EDS_{(4,2)}$ network has [(12 × 6) × 4] meshes, which can be written as 24 × n × (n-1) meshes. Therefore, we can say that in general the network has $\frac{n!(n-1)}{2}$ + (k-1) × 24 × n × (n-1) meshes. After considering all of the above four cases, to get a generalized equation for $EDS_{(n,k)}$, The total number of meshes in $EDS_{(n,k)}$ network will be, $\frac{n!(n-1)}{2}$ +24(k-1)(n)(n-1)(n-2)…3 . ***(Proved)***

By inspecting the Lemmas and Theorem 2 it is clear that the dilation, congestion and expansion of mesh embedding in the EDS topology is unity.

4. Conclusion

In this current study, the Extended Double Star network topology is a unique interconnection system that can be suitable for implementing large-scale parallel computing. As compared to previous networks, this particular EDS network has significantly better qualities such as node degree, diameter, cost, traffic density, and robustness. The architecture is bipartite and Hamiltonian lacerable. Embedding of both the Ring and Mesh topologies can effortlessly done into the Extended Double star network. The longest ring that can be realized in the EDS is estimated. Thus, it is concluded that the EDS contains cycles greater than 6 as subgraphs. This research also finds an upper bound on the number of disjoint meshes that can be embedded into EDS topology. The extended double star network conserves all of the topological features of the original star network. The inclusion of the controller nodes results in faster and economical message passing feature and it will be beneficial for the parallel computing systems. The inner rings computing nodes can also be treated as back up nodes can make the topology more fault tolerant. In addition to that the inner and outer ring design can be a candidate for implementing distributed file structure with map and reduce framework of parallel systems which strongly focuses on scope for some future research. The graph theoretical results of EDS topology obtained here claim that this hierarchical architecture can be very influential from communication and computation point of view in the Big data scenario.

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